ULB-TH/03-34
hep-th/0310083

arXiv:hep-th/0310083v1 8 Oct 2003

Hamiltonian BRST and Batalin-Vilkovisky formalisms for second quantization of gauge theories

Glenn Barnich[*,a] and Maxim Grigoriev[†,a,b]

[a]Physique Théorique et Mathématique,
Université Libre de Bruxelles,
Campus Plaine C.P. 231, B–1050 Bruxelles, Belgium

[b]Tamm Theory Department, Lebedev Physical Institute,
Leninsky prospect 53, 119991 Moscow, Russia

Abstract

Gauge theories that have been first quantized using the Hamiltonian BRST operator formalism are described as classical Hamiltonian BRST systems with a BRST charge of the form $\langle\Psi,\hat{\Omega}\Psi\rangle_{\mathrm{even}}$ and with natural ghost and parity degrees for all fields. The associated proper solution of the classical Batalin-Vilkovisky master equation is constructed from first principles. Both of these formulations can be used as starting points for second quantization. In the case of time reparametrization invariant systems, the relation to the standard $\langle\Psi,\hat{\Omega}\Psi\rangle_{\mathrm{odd}}$ master action is established.

[*] Research Associate of the National Fund for Scientific Research (Belgium).
[†] Postdoctoral Visitor of the National Fund for Scientific Research (Belgium).

Contents

1 Introduction

It has been realized in [1, 2, 3] (see also [4]) that the action of open bosonic string field theory [5, 6], with free part given by the expectation value of the BRST operator, should be understood as a solution to the classical Batalin-Vilkovisky (BV) master equation. The collection of fields, ghosts, and all the ghosts for ghosts corresponds to the coefficients of the states in negative ghost numbers, while the associated antifields correspond to the coefficients of the states in positive ghost numbers. In the standard gauge field theory context, however, a proper solution to the Batalin-Vilkovisky master equation is obtained from a gauge invariant action, a generating set for its non trivial gauge symmetries and, if needed, associated reducibility operators [7, 8, 9, 10, 11] (see also [12, 13] for reviews).

The purpose of this paper is to construct from basic principles the proper solution of the master equation associated to a theory first quantized using the Hamiltonian BRST operator (or "BFV") formalism [14, 15, 16] (see also [17]) and to relate it with the standard master action of [1, 2, 3]. This involves several steps:

1. the reformulation of BRST quantum mechanics as a classical Hamiltonian BRST system;

2. using the known proper solution of the master equation for Hamiltonian BRST systems;

3. for time reparametrization invariant systems, relating the constructed master action to the standard one by showing that they differ by the quantization of classically trivial pairs.

The first two steps are treated in section **3**, while section **4** is devoted to the last step.

More precisely, for the first step, it has been pointed out by many authors (see e.g. [18, 19] and [20, 21, 22] for reviews and further references) that the Hilbert space of quantum mechanics can be understood as a (possibly infinitedimensional) symplectic manifold and that the Schrödinger evolution appears as a Hamiltonian flow on this phase space. This point of view provides a useful set-up for second quantization. In order to apply these ideas to gauge systems quantized in the operator formalism according to the Hamiltonian BRST prescription, one also needs to understand in this context the physical state condition $\hat{\Omega}\psi = 0$, as well as the identification of BRST closed states up to BRST exact ones. The latter two problems alone have been faced in the context of string field theory [6, 5, 4, 23], with the somewhat surprising conclusion that the object $\langle \Psi, \hat{\Omega}\Psi \rangle$ is not a BRST charge, but a solution to the master equation. This is due to the fact that the ghost pair associated to the mass shell constraint is quantized in the Schrödinger representation.

In our approach, we will start by assuming that the number of independent constraints is even so that there is also no fractionalization of the ghost number. There is no loss of generality in this assumption, since one can always include some Lagrange multipliers among the canonical variables together with the constraints that their momenta should vanish.

In subsection **3.1**, we then associate to BRST quantum mechanics a Kähler supermanifold. In particular, the even symplectic form of ghost number 0 is determined by the

imaginary part of the non degenerate hermitian inner product. In appendix **A**, we discuss the geometry of this supermanifold in terms of complex coordinates. In subsection **3.2**, it is shown that, as for non gauge systems, time evolution in the supermanifold corresponds to the Hamiltonian flow determined by the "expectation value" of the BRST invariant Hamiltonian $\mathbf{H} = -\frac{1}{2}\langle\Psi, \hat{H}\Psi\rangle$, where Ψ denotes the "string field". On the supermanifold, the physical state condition then coincides with the constrained surface determined by the zero locus of the BRST charge $\mathbf{\Omega} = -\frac{1}{2}\langle\Psi, \hat{\Omega}\Psi\rangle$. These constraints are first class, and so is **H**. Furthermore, on the supermanifold, the identification of BRST closed states up to BRST exact ones corresponds to considering Dirac observables, i.e., functions defined on the constraint surface that are annihilated by the Poisson bracket with these constraints. As has been shown in [24], constraints associated to the zero locus of the BRST charge are special in the sense that the cohomology of the BRST charge itself provides directly the correct description of these Dirac observables, without the need to further extend the phase space. In order to make the paper self-contained, a formal proof adapted to the BRST charge $\mathbf{\Omega}$ is provided in appendix **B**. From the point of view of the symplectic supermanifold, BRST quantum mechanics becomes thus a classical Hamiltonian BRST system described by **H** and $\mathbf{\Omega}$.

Concerning the second step, the proper solution of the master equation associated to a first order Hamiltonian gauge theory and its relation to the Hamiltonian BRST formalism is well known [25, 26, 27, 28, 29, 30, 31, 32, 33]. A convenient "superfield" reformulation [34] of such a master action also exists. These are reviewed in section **2** together with the basic formulas of BRST operator quantization[1]. In subsection **3.3**, the above results are applied to derive the master action **S** for the classical Hamiltonan BFV system of $\mathbf{\Omega}$ and **H**.

In subsection **4.1**, we discuss tensor products of Hamiltonian BRST quantum mechanical systems at the level of the associated classical field theories. For later use, the assumption that the inner product is even is dropped so that the bracket may be either even or odd. In subsection **4.2**, it is shown that the master action **S** associated to $\mathbf{\Omega}$ and **H** can be directly obtained from the BRST charge $\hat{\Omega}_M$ of the parametrized system: the master action is given by $\mathbf{S} = \frac{1}{2}\langle\Psi_M, \hat{\Omega}_M\Psi_M\rangle_M$, where Ψ_M is the string field of the parametrized system; the ghost pair of the reparametrization constraint is quantized in the Schrödinger representation so that $\langle\cdot,\cdot\rangle_M$ is odd.

Finally, to complete the last step, we consider in subsection **4.3** the case of systems that are already time reparametrization invariant and are quantized with an odd inner product, originating for instance from the Schrödinger representation for the ghosts associated to the mass-shell constraint (see e.g. [35, 36]). The master action **S** is then shown to differ from the original $\mathbf{S}_{\rm st} = \frac{1}{2}\langle\Psi_{\rm st}, \hat{\Omega}\Psi_{\rm st}\rangle_{\rm st}$ by two classically trivial pairs, quantized in the Schödinger representation[2]. More precisely, we show that **S** corresponds to the tensor product of the system described by $\mathbf{S}_{\rm st}$ with the system described by the Hamiltonian BRST charge $\mathbf{\Omega}_{\rm aux}$ associated to the trivial pairs. Had these pairs been quantized in the Fock representation instead, we use the results of subsection **4.1** to show that **S** could have been consistently reduced to $\mathbf{S}_{\rm st}$. In the Schrödinger representation, however,

[1]Except for the conventions related to complex conjugation, we follow closely reference [12], to which we refer for further details.

[2]Trivial pairs in string field theory have been used previously in a different context in [37].

the master action $\mathbf{S}$ involves two more dimensions than $\mathbf{S}_{\rm st}$. In subsection **4.4**, we show that $\mathbf{\Omega}_{\rm aux}$ is the BRST charge of complex Abelian Chern-Simons theory. Without additional ingredients, the master action $\mathbf{S}$ can then not be directly reduced to $\mathbf{S}_{\rm st}$. This is not really surprising since the Fock and the Schrödinger quantization are not unitary equivalent. We conclude by giving some additional remarks on the BRST charge $\mathbf{\Omega}_{\rm aux}$ and the associated master action.

2 Generalities on BFV and BV formalisms

2.1 Classical Hamiltonian BRST theory

A Hamiltonian approach to gauge theories involves a symplectic manifold $\mathcal{M}_0$ with coordinates z^A, constraints G_{a_0}, which we assume for simplicity to be first class and even, $\{G_{a_0}, G_{b_0}\}_{\mathcal{M}_0} = C_{a_0b_0}{}^{c_0}G_{c_0}$, and a first class Hamiltonian H_0 with $\{H_0, G_{a_0}\}_{\mathcal{M}_0} = V_{a_0}{}^{b_0}G_{b_0}$. The constraints may be reducible, $Z^{a_0}_{a_1}G_{a_0} = 0$, with a tower of reducibility equations $Z^{a_{k-1}}_{a_k}Z^{a_{k-2}}_{a_{k-1}} \approx 0$, where $\approx$ means an equality that holds on the constraint surface. Even though we use a finite-dimensional formulation, this section also formally applies to field theories by letting the indices A, a range over both a discrete and a continuous set.

In the Hamiltonian BRST approach, the phase space is extended to a symplectic supermanifold $\mathcal{M}$ by introducing the ghosts η^{a_k} and the ghost momenta $\mathcal{P}_{b_k}$ of parity $k+1$ with $\{\mathcal{P}_{a_k}, \eta^{b_k}\}_{\mathcal{M}} = -\delta^{b_k}_{a_k}$. We take these variables to be real. Our convention for complex conjugation involves transposition of variables together with a minus sign when exchanging two odd variables. On the extended phase space, the ghost number of a function A that is homogeneous in η^a and $\mathcal{P}_b$ is obtained by taking the extended Poisson bracket $\{\,\cdot\,,\,\cdot\,\}_{\mathcal{M}}$ with the purely imaginary function

$$\mathcal{G} = \frac{i}{2}\sum_k (k+1)(\eta^{a_k}\mathcal{P}_{a_k} - \mathcal{P}_{a_k}\eta^{a_k})\,, \qquad \{A, \mathcal{G}\}_{\mathcal{M}} = i\mathrm{gh}(A)\,A\,. \tag{2.1}$$

Out of the contraints, one constructs the nilpotent BRST charge of ghost number 1:

$$\Omega = \eta^{a_0}G_{a_0} + \sum_{k\geq 1}\eta^{a_k}Z^{a_{k-1}}_{a_k}\mathcal{P}_{a_{k-1}} + \dots\,, \qquad \frac{1}{2}\{\Omega, \Omega\}_{\mathcal{M}} = 0. \tag{2.2}$$

Furthermore, the first class Hamiltonian H_0 is extended to the BRST invariant Hamiltonian H of ghost number 0 with $\{H, \Omega\}_{\mathcal{M}} = 0$. Physical quantities such as observables are determined by the BRST cohomology of the differential $s = \{\Omega,\,\cdot\,\}_{\mathcal{M}}$ in the space of functions $F(z, \eta, \mathcal{P})$ on the extended phase space. Time evolution is generated by the BRST invariant Hamiltonian H according to $\dot{F} = \{F, H\}_{\mathcal{M}}$.

2.2 Master action for first order gauge theories

In this subsection, we discuss in some details the proper BV master action for Hamiltonian gauge theories. The reader may wish to skip these details and go directly to the summary, which is the only part that is explicitly needed in the rest of the paper.

The information on the symplectic structure, the dynamics and the constraints of the theory is contained in the extended Hamiltonian action,

$$S_E[z,\lambda] = \int dt\ (\dot{z}^A a_A^{\mathcal{M}_0} - H_0 + \lambda^{a_0} G_{a_0}), \tag{2.3}$$

where λ^{a_0} are Lagrange multipliers. If the symplectic two-form is defined by

$$\sigma_{AB}^{\mathcal{M}_0} = -\frac{\partial^R a_A^{\mathcal{M}_0}}{\partial z^B} - (-1)^{(A+1)(B+1)} \frac{\partial^R a_B^{\mathcal{M}_0}}{\partial z^A}, \tag{2.4}$$

the Poisson bracket is determined by $\left\{z^A, z^B\right\}_{\mathcal{M}_0} = \sigma_{\mathcal{M}_0}^{AB}$ with $\sigma_{\mathcal{M}_0}^{AB}\sigma_{BC}^{\mathcal{M}_0} = \delta_C^A$. Variation with respect to all the fields z^A, λ^{a_0} gives as equations of motions both the dynamical equations and the constraints:

$$\dot{z}^A = \left\{z^A, H_0\right\}_{\mathcal{M}_0}, \qquad G_{a_0} = 0. \tag{2.5}$$

A generating set of gauge symmetries for this action is given by

$$\delta_\epsilon z^A = \epsilon^{a_0}\{G_{a_0}, z^A\}_{\mathcal{M}_0}, \tag{2.6}$$

$$\delta_\epsilon \lambda^{a_0} = \dot{\epsilon}^{a_0} - \lambda^{c_0}\epsilon^{b_0} C_{b_0 c_0}{}^{a_0} - \epsilon^{b_0} V_{b_0}{}^{a_0}, \tag{2.7}$$

for some gauge parameters ϵ^{a_0}

In the field-antifield approach, the functional which contains all the information on the classical action and its gauge algebra is the proper solution S of the classical master equation,

$$\frac{1}{2}(S,S) = 0. \tag{2.8}$$

In the case of the extended Hamiltonian action, the proper solution S is required to start like the original action (2.3), to which one couples through the antifields $z_A^*, \lambda_{a_0}^*$ the gauge transformation (2.6), (2.7) of the fields with the gauge parameters replaced by the ghosts C^{a_0}. One also needs to couple the terms containing the Lagrangian reducibility operators, (which are determined by the Hamiltonian reducibility operators $Z_{a_k}^{a_{k-1}}$) and introduce associated ghosts for ghosts and their antifields. The antifields can be chosen to be real and are defined to be canonically conjugate to the fields with respect to the antibracket $(\cdot,\cdot)$. Additional terms in S are then uniquely determined by the master equation (2.8), up to anticanonical transformations in the antibracket. The proper solution S associated to (2.3) can then be shown to be given by

$$\begin{aligned} S[z,z^*,\lambda,\lambda^*,\eta,\eta^*] = \int dt\ \Big(& \dot{z}^A a_A^{\mathcal{M}_0} + \sum_{k\geq 0} \dot{\eta}^{a_k}\mathcal{P}_{a_k} - H \\ & -z_A^*\left\{z^A,\Omega\right\}_{\mathcal{M}} - \sum_{k\geq 0}[\lambda^{a_k}\left\{\mathcal{P}_{a_k},\Omega\right\}_{\mathcal{M}} + \eta_{a_k}^*\left\{\eta^{a_k},\Omega\right\}_{\mathcal{M}}]\Big), \end{aligned} \tag{2.9}$$

where the identifications $C^{a_k} = \eta^{a_k}$, $C_{a_k}^* = \eta_{a_k}^*$ and $\mathcal{P}_{a_k} = -\lambda_{a_k}^*$ have been made[3]

[3] For later convenience, some signs have been changed in equations (2.3), (2.6) and (2.7) with respect to those of [12]. In (2.9), they imply the change $\Omega \to -\Omega$.

Usually, in order to fix the gauge, one introduces a nonminimal sector, containing antighosts in ghost number -1, their momenta in ghost number 1 and auxiliary fields in ghost number 0. Then, a gauge fixing fermion Υ in ghost number -1 that depends only on the fields is chosen. The choice of Υ is determined by the requirement that there be no more gauge invariance in the dynamics generated by the nonminimal master action obtained after application of the anticanonical transformation generated by Υ and after setting to zero the transformed antifields. This gauge fixed action can be taken as a starting point for a path integral quantization and the partition function can (formally) be shown to be independent of the choice of Υ.

For the master action (2.9), it is possible to fix the gauge without introducing a nonminimal sector: indeed, by considering the anticanonical transformation which consists in the exchange of fields and antifields for the sector of the Lagrange multipliers,

$$(\lambda^{a_k}, \lambda^*_{a_k}) \longrightarrow (-\lambda^*_{a_k}, \lambda^{a_k}) \equiv (\mathcal{P}_{a_k}, \mathcal{P}^{*a_k}), \tag{2.10}$$

the equations of motion are in first order form. The new fields are then the fields $z^\alpha = (z^A, \eta^{a_k}, \mathcal{P}_{a_k})$ that are naturally associated with the Hamiltonian BRST formalism. The antibracket for two functionals $A[z, z^*], B[z, z^*]$ is defined by

$$(A, B)[z, z^*] = \int dt \Big[\frac{\delta^R A}{\delta z^\alpha(t)} \frac{\delta^L B}{\delta z^*_\alpha(t)} - (z^\alpha \longleftrightarrow z^*_\alpha)\Big]. \tag{2.11}$$

The solution S of the master equation (2.5) can be rewritten in a compact way as

$$S[z, z^*] = \int dt \ (\dot{z}^\alpha a^{\mathcal{M}}_\alpha - H - \{z^*_\alpha z^\alpha, \Omega\}_{\mathcal{M}}). \tag{2.12}$$

An additional gauge fixing generated by the fermion $\Upsilon = \int dt \ K(z)$ can then be considered. Its effect is to change the BRST invariant Hamiltonian by a BRST exact term, $H \rightarrow H + \{K, \Omega\}$. Note that after putting to zero the antifields z^*_α, the constraint equations (2.5) are no longer imposed as equations of motions since the associated fields have been put to zero.

In the following, we will not put to zero the antifields obtained after a canonical transformation generated by Υ. This is because during the renormalization process, the antifields allow to conveniently control the Ward identities due to BRST invariance under the form of the Zinn-Justin equation for the effective action. To lowest order in $\hbar$, it is the antifield dependent BRST cohomology of the differential $s = (S, \cdot)$ that controls gauge invariance on the quantum level. This cohomology is invariant under canonical transformations and the introduction of a non minimal sector. Hence, from this point of view, one can forget about gauge fixing and directly discuss the cohomology associated to the master action (2.12). In turn, this cohomology computed in the space of functions in the fields z^α, the antifields z^*_α and their (space)time derivatives can be shown to be isomorphic to the Hamiltonian BRST cohomology of the differential $s_\Omega = \{\Omega, \cdot\}_{\mathcal{M}}$ in the space of functions in z^α (and their spatial derivatives). In the space of local functionals, which is the relevant space in the context of renormalization, the relation between with the Hamiltonian BRST cohomology is more involved [38].

Summary:
From the above construction of the solution (2.12) to the classical master equation, we can learn the following. Suppose that the following data is given:

- a (super) phase space with coordinates z^α and symplectic 2 form generated by $a_\alpha(z)$ with associated Poisson bracket $\{z^\alpha, z^\beta\} = \sigma^{\alpha\beta}_{\mathcal{M}}(z)$,
- a ghost number grading $\mathcal{G}$ on the phase space,
- a nilpotent BRST charge Ω in ghost number 1, whose cohomology determines the physically relevant quantities on the phase space,
- a BRST invariant Hamiltonian H in ghost number 0 determining the time evolution.

Then, in the space of functionals in the fields $z^\alpha(t)$ and additional independent antifields $z^*_\alpha(t)$ of ghost number $-gh(z^\alpha) - 1$ equipped with the antibracket given by (2.11), the proper solution of the master equation is given by (2.12). In order to recover the gauge invariant equations of motion (including the constraints) after putting to zero the antifields, the interpretation of which are the fields and which are the antifields should be reversed for the fields in negative ghost number, $(\mathcal{P}_a, \mathcal{P}^{*a}) \equiv (-\lambda^*_a, \lambda^a) \longrightarrow (\lambda^a, \lambda^*_a)$.

2.3 Superfield reformulation

A superfield reformulation [34] of the master action (2.12) is achieved by introducing an additional Grassmann odd variable θ of ghost number one.

Given an extended phase space $\mathcal{M}$, one associates a space Σ of maps $z^\alpha = z^\alpha_S(t,\theta)$ from the (1|1)-dimensional superspace spanned by t and θ to $\mathcal{M}$. This space is a super-extension of the space of field histories $z^\alpha(t)$ (maps from t to $\mathcal{M}$). Functionals on Σ can be identified with functionals in the fields and antifields of the previous section. Indeed, one can expand $z^\alpha_S(t,\theta)$ into components

$$z^\alpha_S(t,\theta) = z^\alpha(t) + \theta {z^*}_\beta(t)\sigma^{\beta\alpha}_{\mathcal{M}}(z(t))\,, \tag{2.13}$$

which is consistent with the various ghost number assignments. To every functional $\mathcal{A}[z_S]$ one can associate the functional $A[z,z^*]$ obtained by using this expansion. Conversely, to every functional $A[z,z^*]$ corresponds the functional $\mathcal{A}[z_S] = A[\int d\theta\theta z_S, \int d\theta z_S \sigma(z_S)]$.

Functionals on Σ are equipped with the odd Poisson bracket

$$(\mathcal{A},\mathcal{B})[z_S] = (-1)^{|\mathcal{A}|+1}\int dt d\theta \frac{\delta^R \mathcal{A}}{\delta z^\alpha_S(t,\theta)}\sigma^{\alpha\beta}_{\mathcal{M}}(z_S(t,\theta))\frac{\delta^L \mathcal{B}}{\delta z^\beta_S(t,\theta)}, \tag{2.14}$$

with $(\mathcal{A},\mathcal{B})[z+\theta z^*\sigma^{-1}] = (A,B)[z,z^*]$. This Poisson bracket is odd and of ghost number 1. Functional derivatives are defined as

$$\delta\mathcal{A} = \int dt d\theta\, \delta z^\alpha(t,\theta)\frac{\delta^L\mathcal{A}}{\delta z^\alpha(t,\theta)} = \int \frac{\delta^R\mathcal{A}}{\delta z^\alpha(t,\theta)}\delta z^\alpha(t,\theta)\, dt d\theta. \tag{2.15}$$

The master action $\mathcal{S}[z_S]$ corresponding to $S[z, z^*]$ given in (2.12) can then be written as

$$\mathcal{S}[z_S] = \int dtd\theta \left[Dz_S^\alpha a_\alpha^{\mathcal{M}}(z_S) - \theta H(z_S) - \Omega(z_S) \right], \tag{2.16}$$

with $D = \theta \frac{\partial}{\partial t}$.

The superfield reformulation regroups fields and antifields in convenient supermultiplets so that the antibracket is induced by the extended Poisson bracket.

2.4 BRST operator quantization

The BRST operator quantization consists in realizing the functions on the extended phase space as linear operators in a super Hilbert space $\mathcal{H}$ together with the correspondence rule $[\hat{A}, \hat{B}] = i\hbar\widehat{\{A, B\}} + O(\hbar^2)$, where $[\cdot, \cdot]$ denotes the graded commutator and A, B are phase space functions with associated linear operators $\hat{A}, \hat{B}$.

These rules imply in particular that $\frac{1}{2}[\hat{\Omega}, \hat{\Omega}] = O(\hbar^2) = [\hat{H}, \hat{\Omega}]$. In the following, we assume that we are in the non anomalous case, where

$$\frac{1}{2}[\hat{\Omega}, \hat{\Omega}] = 0, \qquad [\hat{H}, \hat{\Omega}] = 0, \tag{2.17}$$

and $\hat{\Omega}$, $\hat{H}$ are hermitian operators in the inner product $\langle \psi, \phi \rangle$, which is non degenerate but not necessarily positive definite and makes the real classical variables hermitian operators. Furthermore, we take $\hbar = 1$. For a super Hilbert space,

$$\overline{\langle \psi, \phi \rangle} = (-1)^{|\psi||\phi|} \langle \phi, \psi \rangle, \tag{2.18}$$

$$\langle \psi, \hat{A}\phi \rangle = (-1)^{|A||\psi|} \langle \hat{A}^\dagger \psi, \phi \rangle, \tag{2.19}$$

where $|\phi|, |A|$ denotes the Grassmann parity of the state, respectively the operator $\hat{A}$. The relation to standard Hilbert space with even and odd elements, for which the above formulas do not involve sign factors, is explained for instance in [39].

In what follows, we are not interested in a probabilistic interpretation of the quantum theory, but rather in an associated classical field theory. This is the reason why we are not concerned here with questions related to the normalizability of states or to the infinite dimensionality of the Hilbert space.

The ghost number of an operator is obtained by taking the graded commutator (from the left) with the antihermitian operator $\hat{\mathcal{G}}$. We assume that $\mathcal{H}$ splits as a sum of eigenstates of $\hat{\mathcal{G}}$, $\mathcal{H} = \oplus_p \mathcal{H}_p$ with $\hat{\mathcal{G}}\psi_p = p\psi_p$ for $\psi_p \in \mathcal{H}_p$. It then follows from the antihermiticity of $\hat{\mathcal{G}}$ that $\langle \psi_p, \phi_{p'} \rangle \neq 0$ only if $p + p' = 0$. This means that the ghost number of the scalar product $\langle , \rangle$ is zero. The ghost number p of a state can be shown to be $p = p_0 + k$ for some integer k with $p_0 = 0$ or $p_0 = \frac{1}{2}$.

The case where $p_0 = \frac{1}{2}$ arises if the number of independent constraints is odd. In this case, one can include some of the Lagrange multipliers λ^a and their momenta b_a among the canonical variables, $\{\lambda^a, b_b\} = \delta^a_b$, together with the new constraint $b_a \approx 0$. On the level of the classical BRST formalism, this implies adding to the extended phase space the antighosts $\bar{C}_a$ of ghost number -1 and their momenta ρ^a of ghost number 1,

$\{\rho^a, \bar{C}_b\} = -\delta^a_b$. All these variables are chosen to be real. The BRST charge of the system is then modified by the addition of the non minimal piece $\Omega^{\rm nm} = \rho^a b_a$. Hence, by adding the cohomologically trivial pairs (λ^a, b_a), $(\rho^a, \bar{C}_a)$, one can always assume $p_0 = 0$, which is what we do unless otherwise specified. We also assume that the inner product is even, $\langle\psi,\, \phi\rangle = 0$ if ψ and ϕ are of opposite parity.

Physical operators are described by hermitian operators $\hat{A}$ such that

$$[\hat{A}, \hat{\Omega}] = 0, \tag{2.20}$$

where two such operators have to be identified if they differ by a BRST exact operator

$$\hat{A} \sim \hat{A} + [\hat{B}, \hat{\Omega}]. \tag{2.21}$$

These two equations define the BRST operator cohomology.

Similarily, physical states are selected by the condition

$$\hat{\Omega}\psi = 0. \tag{2.22}$$

Furthermore, BRST exact states should be considered as zero, or equivalently, states that differ by a BRST exact ones should be identified

$$\psi \sim \psi + \hat{\Omega}\chi. \tag{2.23}$$

These two equations define the BRST state cohomology.

Finally, time evolution is governed by the Schrödinger equation

$$i\frac{d\psi}{dt} = \hat{H}\psi. \tag{2.24}$$

3 BFV and BV formalisms for BRST first quantized gauge systems

3.1 Geometry of BRST quantum mechanics

Let $\{e_a\}$ be a basis over $\mathbb{R}$ of the graded Hilbert space $\mathcal{H}$ such that the basis vectors are of definite Grassmann parity $|a|$ and ghost number ${\rm gh}(e_a)$. A general vector can be written as $\psi = e_a k^a$, with $k^a \in \mathbb{R}$. To each e_a, one associates a real variable Ψ^a of parity $|a|$ and ghost number $-{\rm gh}(e_a)$. These variables are coordinates of a supermanifold $\mathcal{M}_\mathcal{H}$ associated to $\mathcal{H}$. The algebra of real valued functions on this supermanifold is denoted by $\mathfrak{G}$. Introducing the right module $\mathcal{H}_\mathfrak{G} = \mathcal{H} \otimes \mathfrak{G}$ [40], the "string field" appears as the particular element $\Psi = e_a\Psi^a$ of this module. At this stage, it is even and of total ghost number 0.

The sesquilinear form $\langle\cdot,\, \cdot\rangle$ on $\mathcal{H}$ can be extended to elements ψf and ϕg of $\mathcal{H}_\mathfrak{G}$, with $f(\Psi), g(\Psi) \in \mathfrak{G}$ by the rule

$$\langle\psi f,\, \phi g\rangle = (-1)^{|f||\phi|}\langle\psi,\, \phi\rangle f g\,. \tag{3.1}$$

A linear operator $\hat{A}$ on $\mathcal{H}$ is naturally extended to $\mathcal{H}_\mathfrak{G}$: $\hat{A}(\psi f) = (\hat{A}\psi)f$.

The real and imaginary parts of this inner product,

$$\langle\psi,\,\phi\rangle = g(\psi,\phi) + i\omega(\psi,\phi) \tag{3.2}$$

are respectively graded symmetric and graded skew symmetric,

$$g(\psi,\phi) = (-1)^{|\psi||\phi|} g(\phi,\psi)\,, \tag{3.3}$$

$$\omega(\psi,\phi) = -(-1)^{|\psi||\phi|} \omega(\phi,\psi). \tag{3.4}$$

The forms $g(\cdot,\cdot)$ and $\omega(\cdot,\cdot)$ are extended to $\mathcal{H}_{\mathfrak{G}}$ in the same way as $\langle\cdot,\,\cdot\rangle$.

If $\mathcal{H}$ is considered as a superspace over real numbers, both $g(\psi,\phi)$ and $\omega(\psi,\phi)$ are $\mathbb{R}$-bilinear forms on $\mathcal{H}$. The complex structure $\hat{J}$ is the linear operator that represents multiplication by i. As a consequence,

$$g(\hat{J}\phi,\hat{J}\psi) = g(\phi,\psi)\,, \qquad \omega(\hat{J}\phi,\hat{J}\psi) = \omega(\phi,\psi)\,. \tag{3.5}$$

$$g(\hat{J}\phi,\psi) = \omega(\phi,\psi)\,. \tag{3.6}$$

Furthermore, the operator $\hat{J}$ commutes with $\mathbb{C}$-linear operators.

Introducing the coefficients $\omega_{ab} = (-1)^{|a|}\omega(e_a,e_b)$ and defining ω^{ab} through $\omega^{ab}\omega_{bc} = \delta^a_c$, an even graded Poisson bracket on $\mathfrak{G}$ of ghost number 0 is defined by

$$\{f,g\} = \frac{\partial^R f}{\partial\psi^a}\omega^{ab}\frac{\partial^L g}{\partial\psi^b}\,. \tag{3.7}$$

To each antihermitian operator $\hat{A}$, one associates a real quadratic function $F_{\hat{A}}(\Psi) \in \mathfrak{G}$ by

$$F_{\hat{A}}(\Psi) = \frac{1}{2}\langle\Psi,\,-\hat{J}\hat{A}\Psi\rangle\,. \tag{3.8}$$

Antihermiticity implies that $F_{\hat{A}}(\Psi) = \frac{1}{2}\omega(\Psi,\hat{A}\Psi) = -\frac{1}{2}\omega(\hat{A}\Psi,\Psi)$. This map is an homomorphism from the super Lie algebra of antihermitian operators to the super Lie algebra of quadratic real functions in $\mathfrak{G}$ equipped with the Poisson bracket

$$\{F_{\hat{A}},F_{\hat{B}}\} = F_{[\hat{A},\hat{B}]}\,. \tag{3.9}$$

The map is compatible with parity and ghost number assignments, $\mathrm{gh}(F_{\hat{A}}) = \mathrm{gh}(\hat{A})$, $|F_{\hat{A}}| = |\hat{A}|$.

For hermitian operators, we define $\mathbf{A}(\Psi) = F_{-\hat{J}\hat{A}}$. Because of hermiticity

$$\mathbf{A}(\Psi) = -\frac{1}{2}\langle\Psi,\,\hat{A}\Psi\rangle = -\frac{1}{2}g(\Psi,\hat{A}\Psi) = -\frac{1}{2}g(\hat{A}\Psi,\Psi). \tag{3.10}$$

Furthermore, the properties of $\hat{J}$ imply that

$$\{\mathbf{A},\mathbf{B}\} = -\frac{1}{2}\langle\Psi,\,[\hat{A},\hat{B}]\Psi\rangle. \tag{3.11}$$

In particular, for the hermitian BRST charge $\hat{\Omega}$ and the hermitian BRST invariant Hamiltonian $\hat{H}$, equations (2.17) imply

$$\frac{1}{2}\{\mathbf{\Omega},\mathbf{\Omega}\} = 0, \qquad \{\mathbf{H},\mathbf{\Omega}\} = 0, \tag{3.12}$$

where $\mathbf{H},\mathbf{\Omega}$ are of total ghost numbers 0 and 1 respectively.

3.2 BRST quantum mechanics as classical BFV system

The Schrödinger equation in terms of Ψ^a can be written as

$$\frac{d\Psi^a}{dt} = -(\hat{J}\hat{H}\Psi)^a = \{\Psi^a, \mathbf{H}\}, \tag{3.13}$$

so that time evolution of elements $f(\Psi) \in \mathfrak{G}$ is determined by the Hamiltonian flow of $\mathbf{H}$,

$$\frac{df}{dt} = \{f, \mathbf{H}\}. \tag{3.14}$$

On $\mathcal{M}_{\mathcal{H}}$, the physical state condition (2.22) defines a submanifold, the constraint surface determined by

$$\hat{\Omega}^a_b \Psi^b \approx 0. \tag{3.15}$$

Because

$$\{\cdot, \mathbf{\Omega}\} = \frac{\partial^R \cdot}{\partial \Psi^a}(-\hat{J}\hat{\Omega}\Psi)^a \,, \tag{3.16}$$

the constraint surface can be identified with the zero locus of the Hamiltonian vector field associated to $\mathbf{\Omega}$,

$$\mathbf{G}^a \equiv \{\Psi^a, \mathbf{\Omega}\} \approx 0\,. \tag{3.17}$$

By using the graded Jacobi identity for the Poisson bracket and taking (3.12) into account, these constraints are easily shown to be first class,

$$\begin{aligned} \{\mathbf{G}^a, \mathbf{G}^b\} = \{\{\Psi^a, \mathbf{\Omega}\}, \{\Psi^b, \mathbf{\Omega}\}\} &= \{\{\{\Psi^a, \mathbf{\Omega}\}, \Psi^b\}, \mathbf{\Omega}\} \\ &= \frac{\partial^R \{\{\Psi^a, \mathbf{\Omega}\}, \Psi^b\}}{\partial \Psi^c} \mathbf{G}^c \approx 0. \end{aligned} \tag{3.18}$$

Furthermore, since $\mathbf{\Omega}$ is quadratic in Ψ, these constraints are in fact abelian,

$$\{\mathbf{G}^a, \mathbf{G}^b\} = 0. \tag{3.19}$$

On $\mathcal{M}_{\mathcal{H}}$, the identification (2.23) of states up to BRST exact ones, corresponds to taking functions on $\mathfrak{G}$ that are annihilated by the distribution generated by $\hat{\Omega}^a_b \frac{\partial^L \cdot}{\partial \Psi^a}$. This distribution is equivalently generated by the adjoint action of the constraints $\{\mathbf{G}^a, \cdot\}$. The Hamiltonian $\mathbf{H}$ is also first class,

$$\{\mathbf{H}, \mathbf{G}^a\} = \{\{\mathbf{H}, \Psi^a\}, \mathbf{\Omega}\} = \frac{\partial^R \{\mathbf{H}, \Psi^a\}}{\partial \Psi^b} \mathbf{G}^b. \tag{3.20}$$

Hence, from the point of view of $\mathcal{M}_{\mathcal{H}}$, BRST quantum mechanics becomes a classical constraint Hamiltonian system. According to the Dirac theory, an observable is a function $f(\Psi) \in \mathfrak{G}$ such that $\{f, \mathbf{G}^a\} \approx 0$. Two such functions should be considered

equivalent if they coincide on the constraint surface, $f \sim f + \lambda_a \mathbf{G}^a$. Equivalence classes of observables then form a Poisson algebra with respect to the induced bracket.

The classical Hamiltonian BRST approach described in subsection 2.1 consists in extending the phase space in order to encode this Poisson algebra in terms of the cohomology of a BRST charge. This will however not be straightforward in the case of the zero locus constraints $\mathbf{G}^a$, because they are reducible due to the nilpotency of Ω, and for the obvious reducibility operators, they are infinitely reducible.

In fact, it turns out that for the zero locus constraints $\mathbf{G}^a$, there is actually no need to extend the phase space. Indeed, the Poisson algebra of equivalence classes of observables is isomorphic to the cohomology of the BRST charge $\boldsymbol{\Omega}$ itself, equipped with the induced Poisson bracket. This has been shown in [24], where constraint systems originating from the zero locus of a generic Hamiltonian BRST differential have been analyzed. A proof adapted to the particular BRST charge $\boldsymbol{\Omega}$ is given in appendix **B**.

As a side remark, let us note that treating the zero locus of the BRST charge as a constraint surface is analogous to considering the master action S as a classical action; in this case, the zero locus of the BRST differential $s = (S, \cdot)$ is the stationary surface associated to S (see e.g. [41, 42, 24]).

3.3 Proper master action for BRST quantum mechanics

According to subsection 2.2, the solution of the master equation associated to the classical Hamiltonian BRST system on the phase space $\mathcal{M}_\mathcal{H}$ described by $\mathbf{H}$ and $\boldsymbol{\Omega}$ is given by

$$\mathbf{S}[\Psi, \Psi^*] = \int dt\ [\frac{1}{2}\omega(\Psi, \frac{d}{dt}\Psi) - \mathbf{H} - \{\Psi^*_a \Psi^a, \boldsymbol{\Omega}\}] \tag{3.21}$$

which can be written as

$$\mathbf{S}[\Psi, \Psi^*] = \frac{1}{2}\int dt \Big(- i\langle \Psi, \frac{d}{dt}\Psi\rangle + \langle \Psi, \hat{H}\Psi\rangle - \\ - \langle \tilde{\Psi}^*, \hat{\Omega}\Psi\rangle + \langle \Psi, \hat{\Omega}\tilde{\Psi}^*\rangle \Big)\,, \tag{3.22}$$

where $\tilde{\Psi}^{*a} = \Psi^*_b \omega^{ba}$ and $\tilde{\Psi}^* = e_a \tilde{\Psi}^{*a}$. As explained in section 2.2, the role of fields and antifields has been exchanged for those fields that are in strictly negative ghost numbers.

According to subsection 2.3, we now introduce

$$\Psi^a_S(t, \theta) = \Psi^a(t) + \theta \tilde{\Psi}^{*a}(t), \tag{3.23}$$

and also the ghost number 0 object

$$\Psi_S = e_a \Psi^a_S(t, \theta). \tag{3.24}$$

The proper solution (3.22) can then be written as

$$\mathbf{S}[\Psi_S] = \frac{1}{2}\int dt d\theta \Big(- i\theta\langle \Psi_S, \frac{d}{dt}\Psi_S\rangle + \theta\langle \Psi_S, \hat{H}\Psi_S\rangle + \langle \Psi_S, \hat{\Omega}\Psi_S\rangle \Big)\,. \tag{3.25}$$

By construction, it satisfies the master equation with respect to the antibracket (2.13), with $z^A_S(t,\theta)$ replaced by $\Psi^a_S(t,\theta)$,

$$(3.26)\qquad (\mathcal{A},\mathcal{B})[\Psi_S] = (-1)^{|A|+1}\int dtd\theta \frac{\delta^R\mathcal{A}}{\delta\Psi^a_S(t,\theta)}\omega^{ab}\frac{\delta^L\mathcal{B}}{\delta\Psi^b_S(t,\theta)}.$$

For later purposes, it will be useful to rewrite the master action as

$$(3.27)\qquad \mathbf{S}[\Psi_S] = \frac{1}{2}\int dtd\theta\,\langle\Psi_S,(-i\theta\frac{d}{dt}+\theta\hat{H}+\hat{\Omega})\Psi_S\rangle\,.$$

4 Master action and time reparametrization invariance

4.1 Tensor constructions

Given two first quantized BRST systems with super-Hilbert spaces $\mathcal{H}_i$, $i=1,2$, their respective BRST charges $\hat{\Omega}_i$ and Hamiltonians $\hat{H}_i$, the tensor product $\mathcal{H}_T=\mathcal{H}_1\otimes_{\mathbb{C}}\mathcal{H}_2$ is again a super Hilber space. For later use, we do not assume in this section that the inner products on $\mathcal{H}_i$ are even, but we allow them to be of arbitrary parity ε_i. We also admit the possibility of fractionalization of the ghost number. The Grassmann parity and the ghost number of the state $\phi_1\otimes\phi_2$ is naturally $|\phi_1|+|\phi_2|$ and $\mathrm{gh}(\phi_1)+\mathrm{gh}(\phi_2)$ respectively. The inner product on $\mathcal{H}_T$ is determined by

$$(4.1)\qquad \langle\phi_1\otimes\phi_2,\,\psi_1\otimes\psi_2\rangle_T=(-1)^{|\psi_1||\phi_2|}\langle\phi_1,\,\psi_1\rangle_1\langle\phi_2,\,\psi_2\rangle_2\,.$$

It is of parity $\varepsilon_1+\varepsilon_2$ and non degenerate (for $\mathcal{H}_T$ considered as a complex space) provided the ones on $\mathcal{H}_1$ and $\mathcal{H}_2$ are.

Linear operators $\hat{A}_i$ on $\mathcal{H}_i$ determine a linear operator $\hat{A}_T$ on $\mathcal{H}_T$ by

$$(4.2)\quad \hat{A}_T(\phi_1\otimes\psi_2)=(\hat{A}_1\otimes\mathbf{1}+\mathbf{1}\otimes\hat{A}_2)(\phi_1\otimes\psi_2)=(\hat{A}_1\phi_1)\otimes\psi_2+(-1)^{|A_2||\phi_1|}\phi_1\otimes(\hat{A}_2\psi_2)\,.$$

The various definitions imply that

$$(4.3)\qquad \hat{A}_T^\dagger=(\hat{A}_1\otimes\mathbf{1}+\mathbf{1}\otimes\hat{A}_2)^\dagger=\hat{A}_1^\dagger\otimes\mathbf{1}+\mathbf{1}\otimes\hat{A}_2^\dagger\,,$$

and

$$(4.4)\qquad [\hat{A}_T,\hat{B}_T]=[\hat{A}_1,\hat{B}_1]\otimes\mathbf{1}+\mathbf{1}\otimes[\hat{A}_2,\hat{B}_2]\,.$$

In particular, the BRST charges $\hat{\Omega}_i$ and the BRST invariant Hamiltonians $\hat{H}_i$ determine hermitian operators Ω_T and $\hat{H}_T$ such that $\frac{1}{2}[\hat{\Omega}_T,\hat{\Omega}_T]=0$ and $[\hat{H}_T,\hat{\Omega}_T]=0$. Furthermore,

$$(4.5)\qquad H(\hat{\Omega}_T,\mathcal{H}_T)=H(\hat{\Omega}_1,\mathcal{H}_1)\otimes_{\mathbb{C}}H(\hat{\Omega}_2,\mathcal{H}_2).$$

The formal proof is elementary and given in appendix **C**.

If $\{e_\alpha, e_{\bar\alpha}\}$ is a basis of $\mathcal{H}_1^{\mathbb{C}}$ (see appendix **A**), while $\{E_\Lambda, E_{\bar\Lambda}\}$ is a basis of $\mathcal{H}_2^{\mathbb{C}}$, then $\{e_\alpha \otimes E_\Lambda, e_{\bar\alpha} \otimes E_{\bar\Lambda}\}$ is a basis of $\mathcal{H}_T^{\mathbb{C}}$. For these basis vectors, one can consider the complex coordinates $\Psi^{\alpha\Lambda}$ and $\bar\Psi^{\dot\alpha\dot\Lambda}$ for the supermanifold $\mathcal{M}_T$ associated to the superspace $\mathcal{H}_T$ and also the associated complex valued functions $\mathfrak{G}_T^{\mathbb{C}}$. The string fields can now be defined by

$$\Psi_T = (e_\alpha \otimes e_\Lambda)\Psi^{\alpha\Lambda} + (e_{\bar\alpha} \otimes E_{\bar\Lambda})\Psi^{\bar\alpha\bar\Lambda} \,. \tag{4.6}$$

The functions

$$\mathbf{\Omega}_T = -\frac{1}{2}\langle \Psi_T, \hat\Omega_T \Psi_T\rangle_T, \qquad \mathbf{H}_T = -\frac{1}{2}\langle \Psi_T, \hat H_T \Psi_T\rangle_T \,, \tag{4.7}$$

satisfy

$$\frac{1}{2}\{\mathbf{\Omega}_T, \mathbf{\Omega}_T\}_T = 0 = \{\mathbf{H}_T, \mathbf{\Omega}_T\}_T \,, \tag{4.8}$$

where $\{\,\cdot\,,\,\cdot\,\}_T$ denotes a Poisson bracket or antibracket on $\mathfrak{G}_T^{\mathbb{C}}$ determined by imaginary part of $\langle\cdot,\cdot\rangle_T$. Note, however, that when $\langle\cdot,\cdot\rangle_T$ is odd, so is the Poisson bracket $\{\cdot,\cdot\}_{\mathfrak{G}_{\mathbb{C}}^T}$. In this case, it is also called "antibracket" and $\mathbf{\Omega}_T$ is a master action.

If $\{e_\theta, e_{\bar\theta}\}$, $\{E_\Theta, E_{\bar\Theta}\}$ are bases over $\mathbb{C}$ of $H(\hat\Omega_1, \mathcal{H}_1^{\mathbb{C}})$, $H(\hat\Omega_2, \mathcal{H}_2^{\mathbb{C}})$, it follows from appendix **B** and appendix **C** that the cohomology of $\{\cdot, \mathbf{\Omega}_T\}_{\mathfrak{G}_T}$ is isomorphic to real functions in $\Psi^{\theta\Theta}$ and $\Psi^{\bar\theta\bar\Theta}$. In particular, if $H(\hat\Omega_2, \mathcal{H}_2)$ is a one dimensional space over $\mathbb{C}$ with basis vector E such that $\mathrm{gh}(E) = 0, |E| = 0$, then $H(\Omega_T, \mathcal{H}_T) \simeq H(\Omega_1, \mathcal{H}_1)$. It also follows that the cohomology of $\{\cdot, \mathbf{\Omega}_T\}_{\mathfrak{G}_T}$ in $\mathfrak{G}_T$ is isomorphic to that of $\{\cdot, \mathbf{\Omega}_1\}_{\mathfrak{G}_1}$ in $\mathfrak{G}_1$.

Suppose that $\mathcal{H}_2$ contains a quartet or a null doublet. Let $\Lambda = (i, \Lambda')$, where i runs over the states of the quartet or the null doublet. Not only do these states not contribute to the cohomology, but they can also be consistently eliminated from $\mathbf{\Omega}_T$ by reducing the string field used in the construction of $\mathbf{\Omega}_T$ to

$$\Psi'_T = (e_\alpha \otimes E_{\Lambda'})\Psi^{\alpha\Lambda'} + (e_{\bar\alpha} \otimes E_{\bar\Lambda'})\Psi^{\bar\alpha\bar\Lambda'} \,. \tag{4.9}$$

This elimination is algebraic. If the parity of $\langle\cdot,\cdot\rangle_T$ is odd, it corresponds to the elimination of "generalized auxiliary fields" of the master action discussed in [31]. In the case where the parity $\langle\cdot,\cdot\rangle_T$ is even, it is an Hamiltonian analogue of this concept.

4.2 Reinterpretation of master action for BRST quantum mechanics

Consider now the super Hilbert space $\mathcal{H}_{t,\theta}$ obtained by quantizing the phase space $(t, p_0), (\theta, \pi)$ in the Schrödinger representation. The wave functions are $\varphi(t,\theta) = \varphi_0(t) + \theta\varphi_1(t)$, while the inner product is given by

$$\langle \varphi, \varrho\rangle_{t,\theta} = \int dt d\theta\, \bar\varphi(t,\theta)\varrho(t,\theta) \,. \tag{4.10}$$

The ghost number operator is given by $\hat{\mathcal{G}}_{t,\theta} = \theta\dfrac{\partial}{\partial\theta} - \dfrac{1}{2}$ so that $\mathrm{gh}(\varphi_0(t)) = -\frac{1}{2}$, $\mathrm{gh}(\theta\varphi_1(t)) = \frac{1}{2}$ and we take $|\phi_0(t)| = 0$, $|\theta\varphi_1(t)| = 1$ so that the inner product is

Grassmann odd and of ghost number 0. For the super Hilbert space $\mathcal{H}_M = \mathcal{H} \otimes_{\mathbb{C}} \mathcal{H}_{t,\theta}$, where $\mathcal{H}$ is an even super Hilbert space, states are of the form $e_a k_0^a(t) + e_a \theta k_1^a(t)$. Real coordinates on the supermanifold $\mathcal{M}_{\mathcal{H}_M}$ associated to $\mathcal{H}_M$ can be chosen as $k_0^a(t) \to \Psi^a(t)$, $k_1^a(t) \to \tilde{\Psi}^{*a}(t)$. The ghost number 0 object Ψ_S introduced in (3.24) can now be identified with the string field Ψ_M associated to $\mathcal{H}_M$, $\Psi_S \equiv \Psi_M$. The odd inner product on $\mathcal{H}_M$ is denoted by $\langle \cdot, \cdot \rangle_M$ and extended to $\mathcal{H}_M \otimes \mathfrak{G}_M$, where $\mathfrak{G}_M$ is the algebra of real functions in $\Psi^a(t), \tilde{\Psi}^{*a}(t)$. The master action (3.27) can now be written as

$$\mathbf{S}[\Psi_M] = \frac{1}{2}\langle \Psi_M, \hat{\Omega}_M \Psi_M \rangle_M \,, \tag{4.11}$$

$$\hat{\Omega}_M = \hat{\theta}(\hat{p}_0 + \hat{H}) + \hat{\Omega}. \tag{4.12}$$

In this case, the imaginary part of this inner product determines an odd symplectic structure on $\mathcal{M}_{\mathcal{H}_M}$. Its inverse is an antibracket, which coincides with (3.26), up to ghost number assignments discussed below. Only when the BRST invariant Hamiltonian $\hat{H}$ vanishes is $\hat{\Omega}_M$ the tensor product of the BRST charges $\hat{\Omega}$ and $\hat{\theta}\hat{p}_0$.

From the expression of $\hat{\Omega}_M$, we can deduce how to arrive directly at (4.11): make the original classical Hamiltonian system time reparametrization invariant by including the time t and its conjugate momentum p_0 among the canonical variables and adding the first class constraint $p_0 + H_0 \approx 0$. The BRST charge for this system is then given by Ω_M, with (θ, π) the "time reparametrization" ghost pair associated with this new constraint. The quantization of this system then leads directly to $\mathcal{H}_M$ with its odd inner product and the master action (4.11).

From the point of view of $\mathcal{H}_M$, the ghost numbers of fields and antifields are half integer and differ from the standard ones by one half. In particular physical fields are those at ghost number $\frac{1}{2}$). This originates from the additional ghost number operator $\hat{\mathcal{G}}_{t,\theta}$ and our convention for the string field ghost number. Indeed, the fields $\Psi^a(t), \tilde{\Psi}^{*a}$ now have ghost numbers $\frac{1}{2} - \mathrm{gh}(e_a)$ and $-\frac{1}{2} - \mathrm{gh}(e_a)$ instead of $-\mathrm{gh}(e_a)$ and $-1 - \mathrm{gh}(e_a)$, which is the natural assignment from the point of view of the BFV system associated to BRST quantum mechanics, and, as explained in subsection **3.3**, also leads to the standard ghost number assignments in the associated BV formalism. From the point of view of $\mathcal{H}_M$ the standard ghost number assignment for the fields, antifields and the antibracket are thus obtained by shifting the ghost number by $\frac{1}{2}$ so that the inner product carries ghost number -1, while the antibracket is of ghost number 1.

To summarize, we have thus shown that (4.11) is the proper solution of the master equation for a first quantized BRST system defined by $\hat{H}$ and $\hat{\Omega}$ on $\mathcal{H}$. The antibracket is determined by the inverse of the imaginary part of the inner product $\langle \cdot, \cdot \rangle_M$ defined on $\mathcal{H}_M$. Moreover, after shifting, all physical fields are among the fields associated to ghost number zero states while those associated to negative and positive ghost number states are respectively ghost fields and antifields.

4.3 Time reparametrization invariant systems

Suppose now that the BRST invariant Hamiltonian H vanishes, as in time reparametrization invariant systems. Suppose furthermore that the original system has an odd inner product $\langle \cdot, \cdot \rangle_{\mathrm{st}}$. This is the case for instance for the relativistic particle or for the

open bosonic string, where the ghost pair $(\eta, \mathcal{P})$ associated to the mass-shell constraint $p^2 + m^2 \approx 0$, respectively $L_0 \approx 0$, is quantized in the Schrödinger representation. According to our discussion in subsection 2.4, in order to have an even inner product and no fractionalization of the ghost number, the system is extended to include the Lagrange multiplier λ and its momentum b, together with the ghost pair $(\bar{C}, \rho)$ associated to the constraint $b \approx 0$. The BRST charge picks up the additional term $b\rho$ and the pairs (λ, b), $(\bar{C}, \rho)$ are both quantized in the Schrödinger representation, yielding the odd Hilbert space $\mathcal{H}_{\lambda,\bar{C}}$. Hence, the even Hilbert space $\mathcal{H}$ with BRST charge $\hat{\Omega}$ is of the form $\mathcal{H}_{\mathrm{st}} \otimes_{\mathbb{C}} \mathcal{H}_{\lambda,\bar{C}}$, where $\mathcal{H}_{\mathrm{st}}$ is odd with $\hat{\Omega}$ the tensor product of $\hat{\Omega}_{\mathrm{st}}$ and $\hat{b}\hat{\rho}$.

The master action (4.11) can then be understood as resulting from the original system described by the odd Hilbert space $\mathcal{H}_{\mathrm{st}}$, the BRST operator $\hat{\Omega}_{\mathrm{st}}$ and the associated master action

$$\mathbf{S}_{\mathrm{st}} = \frac{1}{2}\langle \Psi_{\mathrm{st}}, \hat{\Omega}_{\mathrm{st}} \Psi_{\mathrm{st}} \rangle_{\mathrm{st}}, \tag{4.13}$$

tensored with the system described by the even Hilbert space $\mathcal{H}_{\mathrm{aux}} = \mathcal{H}_{\lambda,\bar{C}} \otimes_{\mathbb{C}} \mathcal{H}_{t,\theta}$ with inner product $\langle \cdot, \cdot \rangle_{\mathrm{aux}}$, the BRST operator $\hat{\Omega}_{\mathrm{aux}} = \hat{b}\hat{\rho} \otimes \mathbf{1} + \mathbf{1} \otimes \hat{\theta}\hat{p}_0$ and the associated BRST charge

$$\mathbf{\Omega}_{\mathrm{aux}} = -\frac{1}{2}\langle \Psi_{\mathrm{aux}}, \hat{\Omega}_{\mathrm{aux}} \Psi_{\mathrm{aux}} \rangle_{\mathrm{aux}}\,. \tag{4.14}$$

On the classical level, the auxiliary system is described by the pairs $((\lambda, b), (t, p_0))$, the constraints $b \approx 0 \approx p_0$ and the ghost pairs $((\bar{C}, \rho), (\theta, \pi))$. The associated BRST charge $\Omega_{\mathrm{aux}} = \rho b + \theta p_0$ describes 2 trivial pairs and its cohomology is generated by a constant. On the classical level, one can thus simply get rid of these pairs. The question then is whether first quantized BRST systems (and the associated classical field theories) that differ by the quantization of classically trivial pairs are equivalent.

If the 2 pairs had been quantized in the Fock instead of the Schrödinger representation, then equivalence could have been directly established. Indeed, all the states except for the Fock vacuum $|\,0\rangle$ form quartets. Hence, according to the discussion in subsection 4.1, the proper master action (4.14) can be consistently reduced to the master action $\mathbf{S}_{\mathrm{st}}$.

4.4 Quantization of trivial pairs and Chern-Simons

When the 2 pairs are quantized in the Schrödinger representation, it is convenient to rename them as $\sigma^1, p_1, \sigma^2, p_2$, with $[\hat{\sigma}^\alpha, \hat{p}_\beta] = i\delta^\alpha_\beta$. The associated fermionic ghost pairs are $\eta^1, \mathcal{P}_1, \eta^2, \mathcal{P}_2$, with $\mathrm{gh}(\eta^\alpha) = 1$ $\mathrm{gh}(\mathcal{P}_\alpha) = -1$ and $[\hat{\mathcal{P}}_\alpha, \hat{\eta}^\beta] = -i\delta^\beta_\alpha$. Wave functions and inner product are chosen as

$$\langle \phi, \psi \rangle = \int d\sigma^1 d\sigma^2 d\eta^1 d\mathcal{P}_2\, \bar{\phi}(\sigma, \eta^1, \mathcal{P}_2)\psi(\sigma, \eta^1, \mathcal{P}_2) = \\ = \int d\sigma^1 d\sigma^2 d\eta^1 d\mathcal{P}_2\, h_{ij}\phi^i(\sigma, \eta^1, \mathcal{P}_2)\psi^j(\sigma, \eta^1, \mathcal{P}_2)\,, \tag{4.15}$$

where in the second line we have expressed the hermitian inner product in $\mathbb{C}$ in terms of two component real-valued wave functions. The BRST charge and ghost number

operators are given by

$$\hat{\Omega}_{\mathrm{aux}} = -i\eta^1 \frac{\partial}{\partial \sigma_1} - \frac{\partial}{\partial \sigma_2}\frac{\partial}{\partial \mathcal{P}_2}\,, \qquad \hat{\mathcal{G}} = \eta^1 \frac{\partial}{\partial \eta^1} - \mathcal{P}^2 \frac{\partial}{\partial \mathcal{P}^2}\,. \tag{4.16}$$

In particular, states of the form $\psi(x)$, $\eta^1\psi(x)$, $\mathcal{P}_2\psi(x)$, and $\eta^1\mathcal{P}_2\psi(x)$ are respectively of ghost degrees $0, 1, -1$, and 0.

The associated string field is

$$\Psi = \int d^2\sigma\, |\sigma\rangle e_i \Big(\Phi_2^i(\sigma) + \eta^1 P^i(\sigma) + \mathcal{P}_2 D^i(\sigma) + \eta^1 \mathcal{P}_2 \Phi_1^i(\sigma) \Big)\,, \tag{4.17}$$

where the $\Phi_2^i(\sigma), \Phi_1^i(\sigma), P^i(\sigma), D^i(\sigma)$ are the coordinates on the supermanifold associated to the Hilbert space. Their Grassmann parities and ghost numbers are

$$\begin{gathered} |\Phi_2^i| = |\Phi_1^i| = 0, \qquad |P^i| = |D^i| = 1 \\ \mathrm{gh}(\Phi_2^i) = \mathrm{gh}(\Phi_1^i) = 0\,, \quad \mathrm{gh}(P^i) = -1\,, \quad \mathrm{gh}(D^i) = 1\,, \end{gathered} \tag{4.18}$$

so that $\mathrm{gh}(\Psi) = 0, |\Psi| = 0$. From a geometrical point of view, this supermanifold can be understood as the supermanifold of maps from the supermanifold with coordinates $\sigma^\alpha, \eta^1, \mathcal{P}_2$ (configuration space) to $\mathbb{C}$ viewed as a 2-dimensional real space. The Poisson bracket corresponding to the symplectic form $\mathrm{Im}\langle\cdot,\cdot\rangle$ is determined by

$$\big\{\Phi_2^i(\sigma), \Phi_1^j(\sigma')\big\} = -\omega^{ij}\delta(\sigma - \sigma')\,, \qquad \big\{D^i(\sigma), P^j(\sigma')\big\} = \omega^{ij}\delta(\sigma-\sigma')\,. \tag{4.19}$$

Let J^i_j and ω_{ij} denote the complex structure and the symplectic form on $\mathbb{C}$ respectively. Integrating out the Grassmann odd variables $\eta^1, \mathcal{P}_2$, using integrations by parts and redefining the variables as $A_2^i = \Phi_2^i$, $A_1^i = -J^i_j\Phi_1^j$ and $C^i = J^i_j D^j$, the BRST charge becomes

$$\mathbf{\Omega}_{\mathrm{aux}} = -\int d\sigma^1 d\sigma^2\, g_{ij}\left[A_2^i\partial_1 C^j - A_1^i \partial_2 C^j\right]\,. \tag{4.20}$$

In terms of the new variables, the Poisson bracket is determined by

$$\big\{A_1^i(\sigma), A_2^j(\sigma')\big\} = -g^{ij}\delta(\sigma-\sigma')\,, \qquad \big\{C^i(\sigma), P^j(\sigma')\big\} = -g^{ij}\delta(\sigma-\sigma')\,, \tag{4.21}$$

where $g^{ij} = -J^i_k\omega^{kj}$ and satisfies $g_{ij}g^{jk} = \delta^i_k$. The adjoint action $s = \{\,\cdot\,, \mathbf{\Omega}_{\mathrm{aux}}\}$ reads

$$sA^i_\alpha = \partial_\alpha C^i\,, \quad sC^i = 0\,, \quad sP^i = -\partial_1 A_2^i + \partial_2 A_1^i\,. \tag{4.22}$$

From this it follows that the BRST charge $\mathbf{\Omega}_{\mathrm{aux}}$ is the BRST charge of complex Abelian Chern-Simons theory. We conclude by giving some additional remarks on this BRST charge and the associated master action.

Remark 1: Superfield formulation
Introducing the Grassmann odd superfields Λ^i of ghost number 1

$$\Lambda^i(\sigma,\eta) = C^i(\sigma) + \eta^\alpha A^i_\alpha(\sigma) + \eta^2\eta^1 P^i(\sigma), \tag{4.23}$$

the brackets (4.21) are equivalent to

$$\big\{\Lambda^i(\sigma,\eta), \Lambda^j(\sigma',\eta')\big\} = -g^{ij}\delta(\sigma-\sigma')\delta(\eta-\eta')\,. \tag{4.24}$$

The BRST charge (4.20) can then be rewritten as

$$\mathbf{\Omega}_{\mathrm{aux}} = -\frac{1}{2}\int d\sigma^1 d\sigma^2 d\eta^1 d\eta^2\; g_{ij}\Lambda^i(\eta^1\partial_1 + \eta^2\partial_2)\Lambda^j\,. \tag{4.25}$$

Applying the superfield reformulation to get the master action for a BFV system described by the BRST charge $\mathbf{\Omega}$ and vanishing Hamiltonian, one gets

$$\mathbf{S} = \frac{1}{2}\int d\sigma^0 d\sigma^1 d\sigma^2 d\eta^0 d\eta^1 d\eta^2\; g_{ij}\Big(\Lambda^i_S(\eta^0\partial_0 + \eta^1\partial_1 + \eta^2\partial_2)\Lambda^j_S\Big)\,, \tag{4.26}$$

where the time coordinate is denoted by σ^0, the associated Grassmann odd variable by η^0 and Λ^i_S is the superfield depending on σ^μ,η^μ, with $\mu = 0,1,2$.

This coincides with the well-known AKSZ representation [43] of the master action for Abelian Chern Simons theory. The standard formulation can be recovered by identifying η^μ with $d\sigma^\mu$ so that the action takes the form $\mathbf{S} = \frac{1}{2}\int g_{ij}(\Lambda^i_S \wedge d\Lambda^j_S)$, where d is the de Rham differential.

Remark 2: Coordinate representation for the ghosts
If both ghost pairs are quantized in the coordinate representation, one can arrive directly at (4.25) because the superfield (4.23) then appears as the projection of the string field on $\langle\sigma|$. Indeed, the BRST and ghost number operator act on the states as

$$\hat{\Omega} = -i\eta^\alpha\frac{\partial}{\partial\sigma^\alpha}\,, \qquad \hat{\mathcal{G}} = \eta^\alpha\frac{\partial}{\partial\eta_\alpha} - 1\,. \tag{4.27}$$

In particular, states of the form $\psi(\sigma)$, $\eta^\alpha\psi_\alpha(\sigma)$, and $\eta^2\eta^1\chi(\sigma)$ are respectively of ghost degrees $-1, 0$, and 1. In order to have a bosonic field theory, we assign Grassmann parity k mod 2 to the states of ghost degree k. This implies defining the inner product on the super Hilbert space by

$$\langle\phi,\psi\rangle = -i(-1)^{|\psi|}\int d\sigma^1 d\sigma^2 d\eta^1 d\eta^2\; \overline{\phi(\sigma,\eta)}\,\psi(\sigma,\eta)\,, \tag{4.28}$$

so that $\overline{\langle\phi,\psi\rangle} = (-1)^{|\phi||\psi|}\langle\psi,\phi\rangle$. The associated string field of total ghost number and Grassmann parity zero is now

$$\Psi = \int d^2\sigma\,|\sigma\rangle e_i\Lambda^i(\sigma,\eta)\,, \tag{4.29}$$

and the corresponding $\mathbf{\Omega} = -\frac{1}{2}\langle\Psi,\hat{\Omega}\Psi\rangle$ coincides with (4.25).

Remark 3: Non Abelian Chern-Simons theory
In [43], the expression for the master action is derived for the Lie algebra of a compact Lie group. Using the same reasoning as above, it can easily be shown that the associated BRST charge can be compactly written as

$$\boldsymbol{\Omega} = -\frac{1}{2}\int d\sigma^1 d\sigma^2 d\eta^1 d\eta^2 \Big(g_{IJ}\Lambda^I(\eta^1\partial_1 + \eta^2\partial_2)\Lambda^J + \frac{1}{3} f_{IJK}\Lambda^I\Lambda^J\Lambda^K \Big)\,, \tag{4.30}$$

where g_{IJ} denotes the invariant metric and $f_{IJK} = g_{IL} f^L_{JK}$. Put differently, the BRST charge for Chern-Simons theory has exactly the same form as the AKSZ master action (and therefore the classical action). Only the source supermanifolds are different: for the master action, the superdimension is $(3|3)$, while for the BRST charge it is $(2|2)$.

Similar remarks apply for the BRST charge and the master action of the Poisson sigma model [44, 45, 46].

5 Discussion

The new feature of the present paper is the shift of emphasis, on the level appropriate for second quantization, from the master action to the BRST charge. Given a gauge system quantized according to the Hamiltonian BRST approach, one can always make the number of constraints even if necessary. The associated object $\boldsymbol{\Omega} = -\frac{1}{2}\langle\Psi, \hat{\Omega}\Psi\rangle$ is then a nilpotent BRST charge with respect to the even Poisson bracket induced by the imaginary part of the inner product. Out of this BRST charge, the master action can be constructed according to a standard procedure. In particular, for closed string field theory for instance, $\boldsymbol{\Omega}$ is naturally a BRST charge, without the necessity of adding trivial pairs. We plan to discuss this issue in more details in future work.

Acknowledgements

G.B. wants to thank M. Henneaux for suggesting the problem. Useful discussions with I. Batalin, G. Bonelli, L. Houart, A. Semikhatov and I. Tipunin are gratefully acknowledged. The work of GB and MG is supported in part by the "Actions de Recherche Concertées" of the "Direction de la Recherche Scientifique-Communauté Française de Belgique", by a "Pôle d'Attraction Interuniversitaire" (Belgium), by IISN-Belgium, convention 4.4505.86, by the INTAS grant 00-00262, and by the European Commission RTN program HPRN-CT00131, in which the authors are associated to K. U. Leuven. GB is also supported by Proyectos FONDECYT 1970151 and 7960001 (Chile), while MG is supported by RFBR grants 02-01-00930 and 02-01-06096.

Appendix A: Formulation of supermanifold in terms of complex coordinates

The geometrical structures on the supermanifold $\mathcal{M}_{\mathcal{H}}$ can be conveniently expressed in terms of complex coordinates. We follow [47].

Consider the complexification $\mathcal{H}^{\mathbb{C}} = \mathcal{H} \otimes_{\mathbb{R}} \mathbb{C}$ where $\mathcal{H}$ is considered as above as a superspace over $\mathbb{R}$. The complex conjugation of a vector of the form $\alpha\psi,\ \psi \in \mathcal{H},\ \alpha \in \mathbb{C}$ is defined as $\bar{\alpha}\psi$ so that the original Hilbert space $\mathcal{H}$ is a subspace (over $\mathbb{R}$) of vectors satisfying $\bar{\psi} = \psi$.

All $\mathbb{R}$-linear operations on $\mathcal{H}$ can be be extended to $\mathcal{H}^{\mathbb{C}}$ by $\mathbb{C}$-linearity. In particular, $\mathcal{H}^{\mathbb{C}}$ decomposes as $\mathcal{H}^{\mathbb{C}} = \mathcal{H}^{1,0} \oplus \mathcal{H}^{0,1}$ with

$$\hat{J}\psi = i\psi \quad \forall\psi \in \mathcal{H}^{1,0}\,, \qquad \hat{J}\phi = -i\phi \quad \forall\phi \in \mathcal{H}^{0,1}\,. \tag{5.1}$$

Complex conjugation defines a real linear isomorphism between $\mathcal{H}^{1,0}$ and $\mathcal{H}^{0,1}$. Introducing a basis $\{e_\alpha\}$ for $\mathcal{H}^{1,0}$ and $\{e_{\bar{\alpha}}\}$ for $\mathcal{H}^{0,1}$ such that $\overline{e_\alpha} = e_{\bar{\alpha}}$, the inner product $\langle\cdot,\,\cdot\rangle$ extended by $\mathbb{C}$ bi-linearity is determined by

$$\langle e_{\bar{\alpha}}, e_\beta\rangle = (-1)^{|\bar{\alpha}|}h_{\bar{\alpha}\beta}\,, \qquad \langle e_\alpha, e_{\bar{\beta}}\rangle = \langle e_\alpha, e_\beta\rangle = \langle e_{\bar{\alpha}}, e_{\bar{\beta}}\rangle = 0\,. \tag{5.2}$$

The graded-symmetric and graded-antisimmetric components g and ω of $\langle,\rangle$ are determined by

$$g(e_{\bar{\alpha}}, e_\beta) = \frac{1}{2}(-1)^{|\bar{\alpha}|}h_{\bar{\alpha}\beta}\,, \qquad g(e_\alpha, e_{\bar{\beta}}) = \frac{1}{2}(-1)^{|\bar{\beta}|+|\alpha||\bar{\beta}|}h_{\bar{\beta}\alpha}\,, \tag{5.3}$$

$$\omega(e_{\bar{\alpha}}, e_\beta) = \frac{1}{2i}(-1)^{|\bar{\alpha}|}h_{\bar{\alpha}\beta}\,, \qquad \omega(e_\alpha, e_{\bar{\beta}}) = -\frac{1}{2i}(-1)^{|\bar{\beta}|+|\alpha||\bar{\beta}|}h_{\bar{\beta}\alpha}\,, \tag{5.4}$$

with all other components vanishing. From $\langle e_{\bar{\alpha}}, e_\beta\rangle = \langle e_\alpha + e_{\bar{\alpha}}, e_\beta + e_{\bar{\beta}}\rangle$ and the fact that $e_\alpha + e_{\bar{\alpha}}$ is a real vector it follows that

$$\overline{h_{\bar{\alpha}\beta}} = (-1)^{1+(1+|\bar{\alpha}|)(1+|\beta|)}h_{\bar{\beta}\alpha}\,. \tag{5.5}$$

By considering $\mathcal{H}$ as a space over $\mathbb{R}$, $\mathbb{C}$-linear operators are identified with $\mathbb{R}$ linear operators commuting with $\hat{J}$. In the basis $e_\alpha, e_{\bar{\alpha}}$, this means that the matrix of such an operator $\hat{A}$ is block-diagonal with only the diagonal blocks A^α_β and $A^{\bar{\alpha}}_{\bar{\beta}}$ nonvanishing. The fact that $\hat{A}$ is extended from $\mathcal{H}$ to $\mathcal{H}^{\mathbb{C}}$ by $\mathbb{C}$-linearity implies that $\hat{A}$ maps real vectors to real ones so that $\hat{A}(e_\alpha + e_{\bar{\alpha}})$ is again a real vector, which in turn implies that $\overline{A^\alpha_\beta} = A^{\bar{\alpha}}_{\bar{\beta}}$.

Associated to the basis elements $e_\alpha, e_{\bar{\beta}}$, one then introduces variables $\Psi^\alpha, \Psi^{\bar{\alpha}}$ with $|\Psi^\alpha| = |\Psi^{\bar{\alpha}}| = |\alpha|$ and $\mathrm{gh}(\Psi^\alpha) = \mathrm{gh}(\Psi^{\bar{\alpha}}) = -\mathrm{gh}(e_\alpha)$ and considers $\mathfrak{G}^{\mathbb{C}}$, the algebra of complex valued functions in these variables. The complex conjugation in $\mathcal{H}^{\mathbb{C}}$ naturally determines a complex conjugation in $\mathfrak{G}^{\mathbb{C}}$ through $\overline{\Psi^\alpha} = \Psi^{\bar{\alpha}}$ so that the real elements of $\mathfrak{G}^{\mathbb{C}}$ can then be identified with $\mathfrak{G}$. The symplectic form ω on $\mathcal{H}^{\mathbb{C}}$ determines a Poisson bracket in $\mathfrak{G}^{\mathbb{C}}$ determined by

$$\{\Psi^\alpha, \Psi^{\bar{\beta}}\} = 2ih^{\alpha\bar{\beta}}\,, \qquad h_{\bar{\alpha}\beta}h^{\beta\bar{\gamma}} = \delta^{\bar{\alpha}}_{\bar{\gamma}}\,. \tag{5.6}$$

The string field is then given by $\Psi = \Psi^\alpha e_\alpha + \Psi^{\bar{\alpha}}e_{\bar{\alpha}}$.

Appendix B: Dirac observables and cohomology of $\langle\Psi, \hat{\Omega}\Psi\rangle$

Formally, one can assume that a real basis $\{e_a\} \equiv \{e_i, f_m, g_m\}$ in the Hilbert space $\mathcal{H}$ is chosen such that

$$(-\hat{J}\hat{\Omega})e_i = 0\,, \quad (-\hat{J}\hat{\Omega})f_m = g_m\,, \quad (-\hat{J}\hat{\Omega})g_m = 0\,. \tag{B.1}$$

The associated coordinates of the supermanifold are $\{\Psi^i\} \equiv \{\Xi^i, \Upsilon^m, \Phi^m\}$. On the one hand, the differential $\{\cdot, \mathbf{\Omega}\}$ becomes

$$\{\cdot, \mathbf{\Omega}\} = \frac{\partial^R \cdot}{\partial \Phi^m} \Upsilon^m \,, \tag{B.2}$$

so that $H(\{\cdot, \mathbf{\Omega}\})$ is isomorphic to the algebra of functions in Ξ^i alone. On the other hand, the constraints are given by $\Upsilon^m \approx 0$. Antihermiticity and nilpotency of $-\hat{J}\hat{\Omega}$ implies that the symplectic structure in the basis $\{e_i, f_m, g_m\}$ becomes

$$\begin{pmatrix} \omega_{ij} & \omega_{im} & 0 \\ \omega_{kj} & \omega'_{km} & \omega_{kn} \\ 0 & \omega_{lm} & 0 \end{pmatrix} \,, \tag{B.3}$$

with both ω_{ij} and ω_{lm} non degenerate. The inverse has the form

$$\begin{pmatrix} \omega^{ji} & 0 & \tilde{\omega}^{ni} \\ 0 & 0 & \omega^{nr} \\ \tilde{\omega}^{jp} & \omega^{mp} & \tilde{\omega}^{np} \end{pmatrix} \,, \tag{B.4}$$

with both ω^{ji} and ω^{mp} non degenerate. This implies that the adjoint action in the Poisson bracket of the constraints Υ^m generate shifts in the Φ^m, which are thus coordinates along the gauge orbits. Hence, equivalence classes of Dirac observables also correspond to functions in Ξ^i alone.

Appendix C: Quantum BRST state cohomology of tensor products

Let $\{k_\alpha\}$ and $\{K_\Lambda\}$ be bases over $\mathbb{C}$ in $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. Then the vectors $k_{\alpha\Lambda} = k_\alpha \otimes K_\Lambda$ provide a basis (over $\mathbb{C}$) of the tensor product $\mathcal{H}_1 \otimes_{\mathbb{C}} \mathcal{H}_2$. Assume that in the bases $k_\alpha = \{k_\theta, f_\gamma, g_\gamma\}$ and $K_\Lambda = \{K_\Theta, F_\Gamma, G_\Gamma\}$, the BRST charges $\hat{\Omega}_1$ and $\hat{\Omega}_2$ take the Jordan form,

$$\begin{aligned} \hat{\Omega}_1 k_\theta &= 0\,, \quad \hat{\Omega}_1 f_m = g_m\,, \quad \hat{\Omega}_1 g_m = 0\,, \\ \hat{\Omega}_2 K_\Theta &= 0\,, \quad \hat{\Omega}_2 F_M = G_M\,, \quad \hat{\Omega}_2 G_M = 0\,. \end{aligned} \tag{C.1}$$

One then can check that the vectors

$$\begin{gathered} k_{\theta\Theta} = k_\theta \otimes K_\Theta \\ f_{\gamma\Gamma} = f_\gamma \otimes F_\Gamma\,, \quad \tilde{f}_{\gamma\Gamma} = \frac{1}{2}(g_\gamma \otimes F_\Gamma - (-1)^{|f_\gamma|} f_\gamma \otimes G_\Gamma)\,, \\ f^0_{\theta\Gamma} = k_\theta \otimes F_\Gamma\,, \quad f^0_{\gamma\Theta} = f_\gamma \otimes K_\Theta \\ g_{\gamma\Gamma} = g_\gamma \otimes F_\Gamma + (-1)^{|f_\gamma|} f_\gamma \otimes G_\Gamma\,, \quad \tilde{g}_{\gamma\Gamma} = (-1)^{|g_\gamma|} g_\gamma \otimes G_\Gamma\,, \\ g^0_{\theta\Gamma} = (-1)^{|k_\theta|} k_\theta \otimes G_\Gamma\,, \quad g^0_{\gamma\Theta} = g_\gamma \otimes K_\Theta \end{gathered} \tag{C.2}$$

form a Jordan basis for $\hat{\Omega}_T$. Hence, the cohomology of $\hat{\Omega}_T$ is the linear span over $\mathbb{C}$ of $k_{\theta\Theta} = k_\theta \otimes K_\Theta$, so that $H(\hat{\Omega}_T, \mathcal{H}_T) = H(\hat{\Omega}_1, \mathcal{H}_1) \otimes_{\mathbb{C}} H(\hat{\Omega}_2, \mathcal{H}_2)$.

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